An upper bound for the bulk burning rate for systems

Alexander Kiselev and Leonid Ryzhik
Department of Mathematics
University of Chicago
Chicago IL 60637

Abstract

We consider a system of reaction-diffusion equations with passive advection term and Lewis number Le not equal to one. Such systems are used to describe chemical reactions in a flow in a situation where temperature and material diffusivities are not equal. It is expected that the fluid advection will distort the reaction front, increasing the area of reaction and thus speeding up the reaction process. While a variety of estimates on the influence of the flow on reaction are available for a single reaction-diffusion equation (corresponding to the case of Lewis number equal to one), the case of the system is largely open. We prove a general upper bound on the reaction rate in such systems in terms of the reaction rate for a single reaction-diffusion equation, showing that the long time average of reaction rate with $\mathrm{Le} \neq 1$ does not exceed the $\mathrm{Le} = 1$ case. Thus the upper estimates derived for $\mathrm{Le} = 1$ apply to the systems. Both front-like and compact initial data (hot blob) are considered.

1 Introduction

Systems of reaction-diffusion-advection equations describe numerous physical processes that occur when reactants not only diffuse but are also advected by a fluid or wind motion. The simplest situation is when the effect of temperature and concentration variations on the fluid motion may be neglected. Then evolution of temperature $T(t, \mathbf{x})$ and concentration $n(t, \mathbf{x})$ may be described by a system of two equations

$$\begin{aligned} T_t + u \cdot \nabla T &= \kappa \Delta T + \frac{v_0^2}{\kappa} g(T) n \\ n_t + u \cdot \nabla n &= \frac{\kappa}{\mathrm{Le}} \Delta n - \frac{v_0^2}{\kappa} g(T) n, \quad \mathbf{x} = (x, y), \end{aligned} \tag{1}$$

where the velocity u is passive and presumed given. The nonlinearity $g(T)$ is assumed to be of the KPP-type:

$$g(0) = 0, \quad g'(0) \neq 0, \quad g(T) \leq g'(0) T \quad \text{for } T > 0.$$

The Lewis number Le, which is the ratio of temperature and material diffusivities may be different from one. The special case $\mathrm{Le} = 1$ was studied in a great detail in the absence of advection beginning with the pioneering papers of Kolmogorov, Petrovski and Piskunov [15] and Fisher [9]. In this case for the initial data satisfying $T + n = 1$ the system (1) reduces to a single equation for the temperature

$$T_t + u \cdot \nabla T = \kappa \Delta T + \frac{v_0^2}{\kappa} f(T), \tag{2}$$

where $f(T) = g(T)(1 - T)$. Recently the role of advection u in (2) has been a subject of active research (see [19] for a review, and [1, 2, 3, 7, 12, 13, 14, 8] for very recent papers). In particular,

existence of traveling fronts in shear flows [4] and pulsating traveling fronts in periodic flows was established [18, 3]. It was also shown that asymptotically front-like and compactly supported data propagate with the speed of the traveling fronts [3]. The flow may have drastic effect on the front propagation, speeding up the reaction. The physical reason for this phenomenon is believed to be that the advection distorts the front, helping the hot material to warm up the cold one and increasing the area available for reaction. Various estimates for the speed of the traveling front in the presence of advection when Le $= 1$ were obtained in [1, 2, 7, 8, 14, 13], indicating the dependence of the burning enhancement on the intensity and geometry of the flow.

An obvious restriction placed by the model (2) is the assumption Le $= 1$. There are some interesting situations where this is clearly not the case, such as a majority of reactions taking place in viscous liquids, or nuclear combustion in stars, where temperature diffusivity is much higher than material diffusivity and one can assume with a good degree of approximation that Le $= \infty$. It is of interest, therefore, to compare the results already derived for (2) with what one can expect for the system case, where Le $\neq 1$. The problem turns out to be much harder, in particular because of the lack of maximum principle for T. Surprisingly little is known about the system (1) when Le $\neq 1$ even in the absence of advection. While it is known that traveling fronts exist, the set of allowed velocities may differ significantly from the single equation case, at least for some (non-KPP) reactions. A. Bonnet [5] provided examples of reactions for which (2) has a unique traveling front, (1) has two disjoint intervals of possible traveling front velocities. Furthermore, it is not known that temperature T remains uniformly bounded in time. The best known estimate [6] for $\|T(t)\|_\infty$ valid for $u = 0$ grows like $\log\log t$ for large t. The main purpose of this paper is to establish an upper bound on the bulk reaction rate for the full system (1). It shows that the rate of reaction for the system is bounded from above by the traveling front speed for the case Le $= 1$. Therefore Le $\neq 1$ does not provide a speed-up of reaction. This result is established both for front-like and compactly supported initial data. We notice that the papers [7, 13] contain non-trivial upper bounds on the speed of front propagation in some cellular flows and in shear flows respectively. As a corollary of this paper, these bounds extend to the system case.

For the front problem we consider the system (1) in a strip $D = \mathbb{R}_x \times [0, H]_y$. The boundary conditions are periodic in y

$$T(x, y + H) = T(x, y). \tag{3}$$

We assume that initially material on the left is burned, while on the right it is unburned:

$$\begin{aligned} T(0, x, y) = T_0(x, y) \to 1, \quad n(0, x, y) = n_0(x, y) \to 0 \ \text{ as } x \to -\infty \\ T_0(x, y) \to 0, \quad n_0(x, y) \to 1 \ \text{ as } x \to +\infty. \end{aligned} \tag{4}$$

Initially temperature and concentration are equal to zero and one outside a finite interval:

$$n_0(x, y) = 0,\ T_0(x, y) = 1 \ \text{ for } x \le -L_0 \text{ and } n_0(x, y) = 1,\ T_0(x, y) = 0 \text{ for } x \ge L_0 \tag{5}$$

for some $L_0 > 0$, and

$$T_0(x, y) + n_0(x, y) \le C, \quad T_0 \ge 0,\ 1 \ge n_0 \ge 0 \tag{6}$$

in $[-L_0, L_0]$. The flow $u(x, y) \in C^1(\mathbb{R}^2)$ is periodic in x and y:

$$u(x + L, y) = u(x, y), \quad u(x, y + H) = u(x, y) \tag{7}$$

and incompressible:

$$\nabla \cdot u = 0.$$

We assume in addition that it has mean zero:

$$\int_0^L dx \int_0^H dy u(x,y) = 0.$$

We define the reaction rate as

$$V(t) = \int_D T_t(x,y) \frac{dxdy}{H}$$

and its time average as

$$\langle V \rangle_t = \frac{1}{t} \int_0^t V(s) ds. \tag{8}$$

Long-time propagation speed is described by

$$\langle V \rangle_\infty = \limsup_{t \to \infty} \langle V \rangle_t.$$

In a way similar to [7] one may show that, because of the boundary conditions, we have

$$V(t) = -\int_D n_t(x,y) \frac{dxdy}{H} = \frac{v_0^2}{\kappa} \int_D g(T(t,x,y)) n(t,x,y) \frac{dxdy}{H}.$$

It is known [3] that if $\mathrm{Le} = 1$ then there exist pulsating traveling waves of the form $U(x - ct, x, y)$, periodic in the second two variables and monotonic in the first. Such solutions exist for $c \geq c_*$, and solutions with the initial data such as above propagate asymptotically with the minimal speed c_* [10, 11]. Our main result for the front-like data is the following theorem.

Theorem 1 *Let c_* be the minimal speed of the pulsating traveling wave for $Le = 1$. Let $T(t,x,y)$, $n(t,x,y)$ be solution of (1) with periodic boundary conditions (3) and front-like initial data, as in (4), (6), (5), with arbitrary Lewis number Le. Then we have*

$$\langle V \rangle_\infty \leq c_*.$$

Remark. In fact we prove that for any $\varepsilon > 0$, there exists a finite constant C_ε such that for any t,

$$\langle V \rangle_t \leq c_* + \varepsilon + \frac{C_\varepsilon}{t}.$$

Theorem 1 implies that difference in material and thermal diffusivities may not speed up the front propagation relative to the case $\mathrm{Le} = 1$.

Our second result applies to the situation when we have initially an isolated hot spot of material. Let T_0 be compactly supported and n_0 satisfy (6). We no longer impose periodic boundary conditions (3) but rather consider the problem in the whole space. Then the reaction rate is defined by

$$V(t) = \int_{\mathbb{R}^2} T_t(x,y) \frac{dxdy}{H}. \tag{9}$$

Its time average is now scaled not as in (8) but rather as

$$\langle\langle V \rangle\rangle_t = \frac{1}{t^2} \int_0^t V(s) ds$$

and

$$\langle\langle V\rangle\rangle_\infty = \limsup_{t\to\infty}\langle\langle V\rangle\rangle_t,$$

since the total area of an expanding blob of hot material grows as t^2. We may also define a traveling front going in direction $\mathbf{e}$ as a solution of

$$T_t + u\cdot\nabla T = \kappa\Delta T + \frac{v_0^2}{\kappa}g(T)(1-T)$$

of the form $U(\mathbf{e}\cdot\mathbf{x} - ct, \mathbf{x})$ monotonic in the first variable and periodic in the second and such that

$$\lim_{s\to-\infty} U(s,\mathbf{x}) = 1, \quad \lim_{s\to+\infty} U(s,\mathbf{x}) = 0.$$

Such solutions exist for $c \geq c_*(\theta)$ with $c_*(\theta)$ being the minimal speed in direction θ.

Theorem 2 *Let $T(t,x,y)$, $n(t,x,y)$ be solution of (1) with the initial T_0 that is compactly supported and n_0 that satisfies (6). Then the long-time average reaction rate satisfies the upper bound*

$$\langle\langle V\rangle\rangle_\infty \leq \frac{1}{2}\int_0^{2\pi} c_*^2(\theta)d\theta.$$

The first term in the above inequality is equal to the asymptotic bulk reaction rate of a solution of (1) with T_0 having compact support and Le $= 1$. Thus Theorem 2 is the analog of Theorem 1 for such initial data.

Finally, we note that our arguments also provide some information on the important case of ignition-type nonlinearity, where the function $g(T)$ in (1) satisfies $g(T) = 0$ for $0 \leq T \leq T_0$ for some "ignition temperature" $T_0 < 1$. Directly from the proofs, it is clear that in this case we get an upper bound on the reaction rate in terms of the minimal traveling wave velocity of the single equation (2) with any KPP reaction $f(T)$ satisfying $f(T) \geq g(T)(1-T)$.

Theorem 3 *Assume that the reaction function $g(T)$ in (1) is of ignition type. Let $T(t,x,y)$, $n(t,x,y)$ be solution of (1) with periodic boundary conditions (3) and front-like initial data, as in (4), (6), (5), with arbitrary Lewis number Le. Let $f(T)$ be any reaction of KPP type (that is, positive on $(0,1)$ and satisfying $f'(0) > 0$) such that $f(T) \geq g(T)(1-T)$. Let c_* be the minimal speed of the traveling wave in (2) with such $f(T)$. Then we have*

$$\langle V\rangle_\infty \leq c_*$$

Our methods also establish existence of a classical solution to (1) in a way similar to [6] extending the result of that paper to non-zero advection.

We present the proof of Theorem 1 in the rest of the paper. The proof of Theorem 3 follows along the same argument. The main difficulty in the proof lies in the absence of known uniform L^∞ bounds on temperature T. If such bounds were available the proof would be greatly simplified. The proof of Theorem 2 is very similar and hence is omitted.

2 Outline of the proof of Theorem 1

The proof of Theorem 1 will proceed as follows. First we will show that the front cannot move to the right faster than c_*:

Lemma 1 *For any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ and a constant $\lambda_\varepsilon > 0$ such that*

$$T(t, x, y) \leq C_\varepsilon e^{-\lambda_\varepsilon (x-(c_*+\varepsilon)t)}.$$

However, it is not yet known whether $T(t, x, y)$ is uniformly bounded in time. Therefore we may not conclude from Lemma 1 that the conclusion of Theorem 1 holds since a lot of reaction may occur behind the front. To the best of our knowledge the strongest L^∞-bound on T even for $u = 0$ is

$$T(t, x, y) \leq C(1 + \mathrm{loglog}t)$$

obtained in [6]. It suffices for us to establish a weaker upper bound in the presence of non-zero advection.

Lemma 2 *For any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$0 \leq T(t, x, y) \leq C_\varepsilon e^{\varepsilon t}.$$

The proof of Lemma 2 essentially follows the ideas of [6] with some modifications. Lemmas 1 and 2 allow us to prove the analog of Lemma 1 for $n(t, x, y)$:

Lemma 3 *For any $\varepsilon > 0$ there exists a function $\Psi_\varepsilon(\xi) \geq 0$ such that*

$$n(t, x, y) \geq 1 - \Psi_\varepsilon(x - (c_* + \varepsilon)t)$$

and

$$\int_0^\infty \Psi_\varepsilon(\xi) d\xi < \infty.$$

Then Theorem 1 follows easily.

Proof of Theorem 1 from Lemma 3. We first show that Lemma 3 implies Theorem 1. Observe that maximum principle implies that

$$0 \leq n(t, x, y) \leq 1.$$

Then we have

$$\begin{aligned}
\langle V \rangle_t &= \frac{1}{t} \int\limits_D [n_0(t, x, y) - n(t, x, y)] \frac{dxdy}{H} \\
&\leq \frac{1}{t} \int\limits_{-\infty}^{(c_*+\varepsilon)t} dx \int\limits_0^H \frac{dy}{H} n_0(t, x, y) + \frac{1}{t} \int\limits_{(c_*+\varepsilon)t}^{\infty} dx \int\limits_0^H \frac{dy}{H} [1 - n(t, x, y)] \\
&\leq c_* + \varepsilon + \frac{C_\varepsilon}{t}
\end{aligned} \tag{10}$$

and Theorem 1 follows.□

3 Proof of Lemma 1

We now prove Lemma 1. Given any vector $\mathbf{z} \in I\!\!R^2$ define the linear operator

$$L_{\mathbf{z}} = \kappa(\nabla - \mathbf{z})^2 - u \cdot (\nabla - \mathbf{z}) + \frac{v_0^2}{\kappa} g'(0)$$

with periodic boundary conditions. The operator $L_{\mathbf{z}}$ has a unique continuous positive eigenfunction $\phi(x,y)$ corresponding to a simple eigenvalue $\lambda(\mathbf{z})$. The positivity of the eigenfunction is standard, while setting $\phi = e^{\omega}$, it is straightforward to verify that if $u(x,y)$ is mean-zero and incompressible then $\lambda(\mathbf{z}) \geq \frac{v_0^2}{\kappa} g'(0)$. The first expressions for the propagation speed in terms of $\lambda(\mathbf{z})$ were given by Freidlin and Gartner [11, 10]. The most convenient for our purpose representation for the minimal speed of propagation in direction $\mathbf{e}$ was given by Majda and Souganidis [16]:

$$v(\mathbf{e}) = \inf_{z>0} \frac{\lambda(z\mathbf{e})}{z}.$$

This expression also gives the asymptotic speed of propagation in direction $\mathbf{e} = (e_1, e_2)$ for $\mathrm{Le} = 1$ and general initial data satisfying (4), (6), (5). More precisely, we have for such data

$$\lim_{t\to\infty} T(t, c\mathbf{e}t) = \left\{ \begin{array}{ll} 0, & c > v(\mathbf{e}) \\ 1, & c < v(\mathbf{e}) \end{array} \right\}. \tag{11}$$

In particular the minimal speed of pulsating traveling front in direction $\mathbf{e}_1 = (1,0)$, that we denote by c_*, is given by

$$c_* = \inf_{z>0} \frac{\lambda(z\mathbf{e}_1)}{z}.$$

Given a number $z > 0$ let $\Psi(x,y;z)$ be the positive eigenfunction of L_z corresponding to $\lambda(z\mathbf{e}_1)$:

$$\kappa \left[\left(\frac{\partial}{\partial x} - z \right)^2 + \frac{\partial^2}{\partial y^2} \right] \Psi - u \cdot \nabla \Psi + u_1 z \Psi + \frac{v_0^2}{\kappa} g'(0) \Psi = \lambda(z) \Psi.$$

We define a comparison function

$$\phi(t,x,y;z) = C_z e^{-z(x - \lambda(z)t/z)} \Psi(x,y;z)$$

with the constant $C_z > 0$ so large that

$$T_0(x,y) \leq \phi(0,x,y;z).$$

We can find such C since $\Psi(x,y;z) \geq \psi_0 > 0$, and T_0 vanishes for $x \geq L_0$. The function ϕ satisfies a partial differential equation

$$\phi_t + u \cdot \nabla \phi = \kappa \Delta \phi + \frac{v_0^2}{\kappa} g'(0) \phi$$

Recall that $n \leq 1$, and $g(T) \leq g'(0)T$. Therefore T satisfies

$$T_t + u \cdot \nabla T \leq \kappa \Delta T + \frac{v_0^2}{\kappa} g'(0) T$$

and so the maximum principle implies that for all $t > 0$ we have

$$T(t,x,y) \leq \phi(t,x,y;z) = C_z e^{-z(x - \lambda(z)t/z)} \Psi(x,y;z).$$

We choose then $z > 0$ such that $c_* + \varepsilon > \lambda(z)/z$ and obtain the conclusion of Lemma 1 since $\Psi(x,y;z)$ is bounded from above. We remark that the proved bound remains true for any $g(T) \leq MT$ with some $M > 0$ not only KPP type. This observation will extend our arguments to give Theorem 3. $\square$

4 Proof of Lemma 2

Lemma 2 is proved using the technique of [6]. We first prove a local L^p-bound on T.

Lemma 4 *For any $\gamma > 0$ there exists a constant $C > 0$ so that for any unit cube $Q \subset I\!R^2$ we have*

$$\int_Q T^p(t,x,y)dxdy \le Ce^{(\alpha\gamma+\beta\gamma^2)t}$$

with the constants α and β depending only on κ and $U = \|u\|_\infty$.

Proof of Lemma 4. Let $\phi(t,\mathbf{x})$ be a smooth test function and $F(T,n)$ be smooth in both variables. Then we have

$$\frac{\partial}{\partial t}\int_D \phi(t,\mathbf{x})F(T,n)d\mathbf{x} = \int_D \phi_t F d\mathbf{x} \tag{12}$$
$$+\int_D \phi\left[F_T\kappa\Delta T + \frac{v_0^2}{\kappa}F_T g(T)n - F_T u\cdot\nabla T + \frac{\kappa}{\mathrm{Le}}F_n\Delta n - \frac{v_0^2}{\kappa}F_n g(T)n - F_n u\cdot\nabla n\right]d\mathbf{x}$$
$$=\int_D [\phi_t + u\cdot\nabla\phi]Fd\mathbf{x} + \frac{v_0^2}{\kappa}\int_D (F_T - F_n)g(T)nd\mathbf{x} + \kappa\int_D \phi\left[F_T\Delta T + \frac{F_n}{\mathrm{Le}}\Delta n\right]d\mathbf{x}$$

We rewrite the last expression above as follows:

$$\int_D \phi\left[F_T\Delta T + \frac{F_n}{\mathrm{Le}}\Delta n\right]d\mathbf{x} = -\int_D \left[F_T\nabla\phi\cdot\nabla T + \frac{F_n}{\mathrm{Le}}\nabla\phi\cdot\nabla n\right]d\mathbf{x}$$
$$-\int_D \phi\left[F_{TT}|\nabla T|^2 + F_{Tn}\nabla n\cdot\nabla T + \frac{F_{nn}}{\mathrm{Le}}|\nabla n|^2 + \frac{F_{nT}}{\mathrm{Le}}\nabla n\cdot\nabla T\right]d\mathbf{x}$$

and insert it into (12) to get

$$\frac{\partial}{\partial t}\int_D \phi F d\mathbf{x} = \int_D [\phi_t + u\cdot\nabla\phi + \kappa\Delta\phi]\,Fd\mathbf{x} + \kappa\left(1-\frac{1}{\mathrm{Le}}\right)\int_D F_n\nabla\phi\cdot\nabla n d\mathbf{x} \tag{13}$$
$$-\kappa\int_D \phi\left[\frac{F_{nn}}{\mathrm{Le}}|\nabla n|^2 + \left(1+\frac{1}{\mathrm{Le}}\right)F_{Tn}\nabla n\cdot\nabla T + F_{TT}|\nabla T|^2\right]d\mathbf{x} - \frac{v_0^2}{\kappa}\int_D [F_n - F_T]g(T)nd\mathbf{x}.$$

We choose F in such a way that

$$F_n \ge 2F_T \tag{14}$$

and

$$\left(1+\frac{1}{\mathrm{Le}}\right)^2 F_{nT}^2 \le F_{nn}F_{TT}. \tag{15}$$

Namely, we can take

$$F = (A + n + n^2)e^{\varepsilon T}$$

with A and ε satisfying

$$2\varepsilon(A+2) \le 1, \quad A > 5\left(1+\frac{1}{\mathrm{Le}}\right)^2.$$

Then (14) and (15) hold and we obtain from (13):

$$\frac{\partial}{\partial t}\int\limits_D \phi F d\mathbf{x} \leq \int\limits_D \left[\phi_t + u\cdot\nabla\phi + \kappa\Delta\phi\right] F d\mathbf{x} + \kappa\left(1-\frac{1}{\mathrm{Le}}\right)\int\limits_D F_n \nabla\phi\cdot\nabla n d\mathbf{x} \tag{16}$$

$$-\frac{\kappa}{2}\int\limits_D \phi\left[\frac{F_{nn}}{\mathrm{Le}}|\nabla n|^2 + F_{TT}|\nabla T|^2\right] d\mathbf{x} - \frac{v_0^2}{2\kappa}\int\limits_D F_n g(T) n d\mathbf{x}.$$

We choose the function ϕ of the form

$$\phi(\mathbf{x}) = \frac{1}{(1+\gamma^2|\mathbf{x}-\mathbf{x}_0|^2)^2},$$

then

$$|\nabla\phi| \leq 2\gamma\phi, \quad |\Delta\phi| \leq 20\gamma^2\phi.$$

Let us also denote $U = \|u\|_\infty$, then we get from (16) with an appropriate constant $C > 0$:

$$\frac{\partial}{\partial t}\int\limits_D \phi F d\mathbf{x} \leq C\int\limits_D \left[U\gamma\phi + \kappa\gamma^2\phi\right] F d\mathbf{x} + 2\kappa\left(1-\frac{1}{\mathrm{Le}}\right)\int\limits_D F_n\gamma\phi|\nabla n| d\mathbf{x} - \frac{\kappa}{2}\int\limits_D \phi\frac{F_{nn}}{\mathrm{Le}}|\nabla n|^2 d\mathbf{x}$$

$$\leq \left(CU\gamma + \kappa\gamma^2\right)\int\limits_D \phi F d\mathbf{x} + 2\kappa\mathrm{Le}\left(1-\frac{1}{\mathrm{Le}}\right)^2\gamma^2\int\limits_D \phi\frac{F_n^2}{F_{nn}} d\mathbf{x}. \tag{17}$$

(in the last step we replaced the quadratic expression involving $|\nabla n|$ by its maximum). Observe that

$$\frac{F_n^2}{F_{nn}} = \frac{(1+2n)^2 e^{2\varepsilon T}}{2e^{\varepsilon T}} \leq 5e^{\varepsilon T} \leq 2F$$

as long as $A \geq 3$. Then we have

$$\frac{\partial}{\partial t}\int\limits_D \phi F d\mathbf{x} \leq \sigma\int\limits_D \phi F d\mathbf{x} \tag{18}$$

with

$$\sigma = C\left[U\gamma + \kappa\gamma^2\right] + 4\kappa\left(1-\frac{1}{\mathrm{Le}}\right)^2\gamma^2$$

Therefore we obtain

$$\int\limits_D \phi F d\mathbf{x} \leq \Gamma e^{\sigma t}$$

with

$$\Gamma = \int\limits_D \phi(\mathbf{x}) F(T_0(\mathbf{x}), n_0(\mathbf{x})) d\mathbf{x} \leq \frac{\pi e(A+2)}{\gamma^2}.$$

Then we have for any unit cube Q and any positive integer k:

$$\Gamma e^{\sigma t} \geq \int\limits_D \phi F d\mathbf{x} \geq A\int\limits_D \phi e^{\varepsilon T} d\mathbf{x} \geq \frac{A\varepsilon^k}{k!}\int\limits_Q \phi(\mathbf{x}) T^k d\mathbf{x} \geq \frac{A\varepsilon^k}{k!(1+\gamma^2)^2}\int\limits_Q T^k d\mathbf{x}.$$

Here we have chosen $\mathbf{x}_0$ in the definition of ϕ to be the center of the cube Q. Therefore we obtain

$$\int_Q T^k d\mathbf{x} \leq A^{-1}\Gamma e^{\sigma t}(1+\gamma^2)\varepsilon^{-k} k!$$

for positive integers k. Using interpolation we get for all $p > 1$

$$\int_Q T^p d\mathbf{x} \le A^{-1}\Gamma e^{\sigma t}(1+\gamma^2)\varepsilon^{-p}(p+1)^{p+1}.$$

This finishes the proof of Lemma 4.□

We prove now Lemma 2 using Lemma 4. Let $G(t, \mathbf{x}; \mathbf{z})$ be the Green's function of the advection-diffusion equation

$$\phi_t + u \cdot \nabla \phi = \kappa \Delta \phi$$

posed in the whole space $I\!R^2$. Then we extend T_0 periodically to the whole space $I\!R^2$ from D and obtain

$$\begin{aligned} T(t,\mathbf{x}) &= \int\limits_{I\!R^2} G(t,\mathbf{x};\mathbf{z})T_0(\mathbf{z})d\mathbf{z} + \int\limits_0^t ds \int\limits_{I\!R^2} G(t-s,\mathbf{x};\mathbf{z})g(T(s,\mathbf{z}))n(s,\mathbf{z})d\mathbf{z} \\ &\le 1 + \int\limits_0^t ds \int\limits_{I\!R^2} G(t-s,\mathbf{x};\mathbf{z})g(T(s,\mathbf{z}))n(s,\mathbf{z})d\mathbf{z}. \end{aligned}$$

The Green's function satisfies a uniform upper bound [17]

$$G(t,\mathbf{x};\mathbf{z}) \le \frac{C}{4\pi\nu t} e^{-|\mathbf{x}-\mathbf{z}|^2/(4\nu t)} = \tilde{G}(t,\mathbf{x}-\mathbf{z}) \tag{19}$$

with some positive constants C and ν that depend on κ and $u(x,y)$. Then we get

$$\begin{aligned} T(t,\mathbf{x}) &\le 1 + g'(0)\int\limits_0^t ds \int\limits_{I\!R^2} \tilde{G}(t-s,\mathbf{x}-\mathbf{z})T(s,\mathbf{z})n(s,\mathbf{z})d\mathbf{z} \\ &= 1 + \frac{Cg'(0)}{4\pi\nu t}\int\limits_0^t ds \int\limits_{I\!R^2} e^{-|\mathbf{x}-\mathbf{z}|^2/(4\nu(t-s))}T(s,\mathbf{z})n(s,\mathbf{z})d\mathbf{z}. \end{aligned}$$

We split the last integral into sum of integrals over unit cubes Q_j with the cube Q_0 centered at the point $\mathbf{x}$:

$$\begin{aligned} &\int\limits_{I\!R^2} e^{-|\mathbf{x}-\mathbf{z}|^2/(4\nu(t-s))}T(s,\mathbf{z})n(s,\mathbf{z})d\mathbf{z} = \sum_j \int\limits_{Q_j} e^{-|\mathbf{x}-\mathbf{z}|^2/(4\nu(t-s))}T(s,\mathbf{z})n(s,\mathbf{z})d\mathbf{z} \\ &\le \sum_j e^{-(\mathrm{dist}(\mathbf{x},Q_j))^2/(8\nu(t-s))} \int\limits_{Q_j} e^{-|\mathbf{x}-\mathbf{z}|^2/(8\nu(t-s))}T(s,\mathbf{z})n(s,\mathbf{z})d\mathbf{z} \end{aligned}$$

We use the Hölder inequality with $p > 1$ in the integral above:

$$\begin{aligned} &\int\limits_{Q_j} e^{-|\mathbf{x}-\mathbf{z}|^2/(8\nu(t-s))}T(s,\mathbf{z})n(s,\mathbf{z})d\mathbf{z} \le \left[\int\limits_{I\!R^2} e^{-q|\mathbf{x}-\mathbf{z}|^2/(8\nu(t-s))}d\mathbf{z}\right]^{1/q} \left[\int\limits_{Q_j} T^p(s,\mathbf{z})n^p(s,\mathbf{z})d\mathbf{z}\right]^{1/p} \\ &\le \left[\frac{8\pi\nu(t-s)}{q}\right]^{1/q} \left[\int\limits_{Q_j} T^p(s,\mathbf{z})d\mathbf{z}\right]^{1/p} \le C(\gamma,\nu,q)\mathrm{Le}(t-s)^{1/q}e^{(\alpha\gamma+\beta\gamma^2)t/p} \end{aligned}$$

Therefore we have

$$\int_{I\!R^2} \tilde{G}(t-s,\mathbf{x}-\mathbf{z})T(s,\mathbf{z})n(s,\mathbf{z})d\mathbf{z} \le \frac{C}{(4\pi\nu(t-s))}(t-s)^{1/q}e^{(\alpha\gamma+\beta\gamma^2)t/p}\sum_j e^{-(\mathrm{dist}(\mathbf{x},Q_j)^2/(8\nu(t-s))}$$

Note that if $\mathbf{z}\in Q_j$ with $j\neq 0$ then

$$|\mathbf{x}-\mathbf{z}|\le 2\mathrm{dist}(\mathbf{x},Q_j)$$

and therefore

$$\begin{aligned}
&\int_{I\!R^2} \tilde{G}(t-s,\mathbf{x}-\mathbf{z})T(s,\mathbf{z})n(s,\mathbf{z})d\mathbf{z} \le C(t-s)^{-1/p}e^{(\alpha\gamma+\beta\gamma^2)t/p}\left[1+\int_{I\!R^2} e^{-|\mathbf{x}-\mathbf{z}|^2/(16\nu(t-s))}d\mathbf{z}\right] \\
&\le C(t-s)^{-1/p}e^{(\alpha\gamma+\beta\gamma^2)t/p}\left[1+C(t-s)\right] \le Ce^{(\alpha\gamma+\beta\gamma^2)t/p}\left[(t-s)^{-1/p}+(t-s)^{1/q}\right].
\end{aligned}$$

Finally integrating over $s\in[0,t]$ we obtain

$$T(t,\mathbf{x})\le 1+Ce^{(\alpha\gamma+\beta\gamma^2)t/p}\left[t^{1/q}+t^{1+1/q}\right]\le C'e^{(\alpha\gamma+\beta\gamma^2)t/p}.$$

However, $\gamma>0$ is arbitrary and thus Lemma 2 follows. □

5 Proof of Lemma 3.

Let us define $W=1-C$. It satisfies a differential equation

$$W_t+u\cdot\nabla W=\frac{\kappa}{\mathrm{Le}}\Delta W+\frac{v_0^2}{\kappa}g(T)n.$$

Lemmas 1 and 2 imply that given any $\varepsilon>0$ we may find $C_\varepsilon,\lambda_\varepsilon>0$ such that

$$T(t,x,y)\le C_\varepsilon e^{\varepsilon t},\quad T(t,x,y)\le C_\varepsilon e^{-\lambda_\varepsilon(x-c_\varepsilon t)}$$

with $c_\varepsilon=c_*+\varepsilon$. Let us define

$$R(t,x)=\frac{v_0^2}{\kappa}\min\left(C_\varepsilon e^{\varepsilon t},C_\varepsilon e^{-\lambda_\varepsilon(x-c_\varepsilon t)}\right).$$

Then we have by the maximum principle

$$W(t,x,y)\le\Phi(t,x,y)$$

with the function Φ satisfying the initial value problem

$$\begin{aligned}
&\Phi_t+u\cdot\nabla\Phi=\frac{\kappa}{\mathrm{Le}}\Delta\Phi+R(t,x)\\
&\Phi(0,x,y)=T_0(x,y)\le H(-x+L_0).
\end{aligned}$$

Here $H(x)$ is the Heaviside function:

$$H(x)=\begin{cases}0, x\le 0\\ 1, x>0\end{cases}$$

and L_0 is as in (5). In the sequel we will assume without loss of generality that $L_0 = 0$. We let $\Gamma(t, \mathbf{x}; \mathbf{x}')$ be the fundamental solution of

$$\phi_t + u \cdot \nabla\phi = \frac{\kappa}{\mathrm{Le}}\Delta\phi$$

and obtain ($\mathbf{x}' = (x', y')$)

$$\Phi(t, \mathbf{x}) = \int_{I\!R^2} \Gamma(t, \mathbf{x}; \mathbf{x}')H(-x')d\mathbf{x}' + \int_0^t ds \int_{I\!R^2} \Gamma(t-s, \mathbf{x}; \mathbf{x}')R(s, x') = \Phi_1 + \Phi_2.$$

We will bound Φ_1 and Φ_2 separately. We have a bound for $\Gamma(t, \mathbf{x}; \mathbf{x}')$ similar to (19):

$$\Gamma(t, \mathbf{x}; \mathbf{x}') \leq \frac{C}{\kappa t} e^{-|\mathbf{x}-\mathbf{x}'|^2/(C\kappa t)} \tag{20}$$

with the constant C depending on κ, u and Le. Then we obtain

$$\Phi_1(t, \mathbf{x}) \leq \frac{C}{\kappa t} \int_{I\!R^2} e^{-[(x-x')^2+(y-y')^2]/(C\kappa t)} H(-x')dx'dy' = C' \int_{-\infty}^{-x/\sqrt{C\kappa t}} e^{-z^2} dz \leq Ce^{-x^2/(C\kappa t)}$$

for $x > 0$. Observe that given any $C > 0$ we may choose $\lambda_0 > 0$ and $B > 0$ so that for all $x > 0$ and all $t > 0$ we have

$$e^{-x^2/(C\kappa t)} \leq Be^{-\lambda_0(x-c_\varepsilon t)}. \tag{21}$$

Indeed we need to find λ_0 such that

$$\min_{x>0,t>0} \left\{ \frac{x^2}{C\kappa t} - \lambda_0 x + \lambda_0 c_\varepsilon t \right\} > -\infty.$$

However, we have for $x > 0$, $t > 0$

$$\frac{x^2}{Ct} - \lambda_0 x + \lambda_0 c_\varepsilon t \geq \frac{2x\sqrt{\lambda_0 c_\varepsilon}}{\sqrt{C}} - \lambda_0 x > 0$$

for λ_0 sufficiently small. Therefore (21) holds for such λ_0 and thus we have

$$\Phi_1(t, \mathbf{x}) \leq Ce^{-\lambda_0(x-c_\varepsilon t)} \text{ for } x > 0. \tag{22}$$

Our next goal is to obtain such bound for Φ_2. We use the inequality (20) to get

$$\Phi_2(t, \mathbf{x}) \leq C \int_0^t ds \int_{I\!R} \frac{dx'}{\sqrt{\kappa(t-s)}} e^{-(x-x')^2/(C\kappa(t-s))} R(s, x'). \tag{23}$$

Recall that $R(t, x)$ is defined by

$$R(t, x) = \begin{cases} C_\varepsilon e^{\varepsilon t}, & x \leq X(t) \\ C_\varepsilon e^{-\lambda_\varepsilon(x-c_\varepsilon t)}, & x > X(t) \end{cases}.$$

Here $X(t)$ is defined by

$$\varepsilon t = -\lambda_\varepsilon(X(t) - c_\varepsilon t), \quad X(t) = \left(c_\varepsilon - \frac{\varepsilon}{\lambda_\varepsilon}\right) t. \tag{24}$$

We split the integral in (23) accordingly:

$$\Phi_2(t,\mathbf{x}) \le C'_\varepsilon \int\limits_0^t ds \int\limits_{-\infty}^{X(s)} \frac{dx'}{\sqrt{\kappa(t-s)}} e^{-(x-x')^2/(C\kappa(t-s))} e^{\varepsilon s} \tag{25}$$

$$+C'_\varepsilon \int\limits_0^t ds \int\limits_{X(s)}^{\infty} \frac{dx'}{\sqrt{\kappa(t-s)}} e^{-(x-x')^2/(C\kappa(t-s))-\lambda_\varepsilon(x'-c_\varepsilon s)} = \Phi_{21}+\Phi_{22}.$$

The term Φ_{21} is bounded as follows:

$$\Phi_{21}(t,\mathbf{x}) \le C_\varepsilon \int\limits_0^t ds \int\limits_{-\infty}^{(X(s)-x)/\sqrt{C\kappa(t-s)}} dx' e^{-x'^2} e^{\varepsilon s}$$

Recall that for $\alpha<0$ we have $\int_{-\infty}^{\alpha} e^{-x^2}dx \le e^{-\alpha^2}$. Therefore we have for $x>c_\varepsilon t$:

$$\Phi_{21}(t,\mathbf{x}) \le C_\varepsilon \int\limits_0^t ds e^{\varepsilon s-(X(s)-x)^2/(C\kappa(t-s))} \le C_\varepsilon \int\limits_0^t ds e^{\varepsilon s-(x-c_\varepsilon s)^2/(C\kappa(t-s))}.$$

We use now the inequality (21) to get

$$\Phi_{21}(t,\mathbf{x}) \le C_\varepsilon \int\limits_0^t ds e^{\varepsilon s-\lambda_0(x-c_\varepsilon s-c_\varepsilon(t-s))} \le C_\varepsilon e^{-\lambda_0(x-(c_\varepsilon+\varepsilon/\lambda_0)t)}. \tag{26}$$

It remains to bound Φ_{22}. The function Φ_{22} solves the initial value problem

$$\frac{\partial \Phi_{22}}{\partial t} = C\kappa \frac{\partial^2 \Phi_{22}}{\partial x^2} + C_2 e^{-\lambda_\varepsilon(x-c_\varepsilon t)} H(x-X(t))$$
$$\Phi_{22}(0,x)=0$$

with $H(x)$ being the Heaviside function. However, we have using (24)

$$e^{-\lambda_\varepsilon(x-c_\varepsilon t)} H(x-X(t)) \le e^{\varepsilon t} H(-x+c_\varepsilon t) + e^{-\lambda_\varepsilon(x-c_\varepsilon t)} H(x-c_\varepsilon t),$$

and therefore

$$\Phi_{22} \le \Phi_{23}+\Phi_{24}.$$

The function Φ_{23} solves the initial value problem

$$\frac{\partial \Phi_{23}}{\partial t} = C\kappa \frac{\partial^2 \Phi_{23}}{\partial x^2} + C_2 e^{\varepsilon t} H(-x+c_\varepsilon t)$$
$$\Phi_{23}(0,x)=0.$$

It is bounded exactly in the same way as Φ_{21} but with $X(t)$ replaced by $x(t)=c_\varepsilon t$. This gives an upper bound

$$\Phi_{23}(t,\mathbf{x}) \le C_\varepsilon e^{-\lambda_0(x-(c_\varepsilon+\varepsilon/\lambda_0)t)}. \tag{27}$$

The function Φ_{24} solves the initial value problem

$$\frac{\partial \Phi_{24}}{\partial t} = C\kappa \frac{\partial^2 \Phi_{24}}{\partial x^2} + C_2 e^{-\lambda_\varepsilon(x-c_\varepsilon t)} H(x-c_\varepsilon t)$$
$$\Phi_{24}(0,x)=0.$$

It may be bounded from above by

$$\Phi_{24}(t,x) \leq \Psi(x - c_\varepsilon t).$$

Here $\Psi(\xi)$ is a positive solution of

$$-c_\varepsilon \Psi' = C\kappa \Psi'' + C_2 e^{-\lambda_\varepsilon \xi} H(\xi).$$

A general solution of the above ODE is given by

$$\Psi(\xi) = \Psi_0 + \frac{C\kappa}{c_\varepsilon}\Psi_1 \left[1 - e^{-c_\varepsilon \xi/(C\kappa)}\right], \quad \text{for } \xi < 0$$

and for $\xi > 0$

$$\Psi(\xi) = \Psi_0 + \frac{C\kappa}{c_\varepsilon}\Psi_1 \left[1 - e^{-c_\varepsilon \xi/(C\kappa)}\right] + \frac{C_2}{(\lambda_\varepsilon(c_\varepsilon - \lambda_\varepsilon C\kappa)}\left[e^{-\lambda_\varepsilon \xi} - e^{-c_\varepsilon \xi/C\kappa}\right] + \frac{C_2}{\lambda_\varepsilon C\kappa}\left[1 - e^{-c_\varepsilon \xi/C\kappa}\right].$$

Let us require that

$$\Psi_0 + \frac{C\kappa}{c_\varepsilon}\Psi_1 + \frac{C_2}{\lambda_\varepsilon c_\varepsilon} = 0.$$

Then the function Ψ takes the form

$$\Psi(\xi) = \Psi_0 e^{-c_\varepsilon \xi/(C\kappa)} + \frac{C_2}{\lambda_\varepsilon c_\varepsilon}\left(e^{-c_\varepsilon \xi/(C\kappa)} - 1\right)$$

for $\xi < 0$, and

$$\Psi(\xi) = \left[\Psi_0 - \frac{C_2}{\lambda_\varepsilon(c_\varepsilon - C\kappa\lambda_\varepsilon)}\right] e^{-c_\varepsilon \xi/(C\kappa)} + \frac{C_2}{\lambda_\varepsilon(c_\varepsilon - C\kappa\lambda_\varepsilon)} e^{-\lambda_\varepsilon \xi}$$

for $\xi > 0$ (we can always assume that the denominator is nonzero by shifting λ_ε a little bit). Therefore $\Psi > 0$ as long as Ψ_0 is non-negative, in particular we may choose $\Psi_0 = 0$. Then we have

$$\Phi_{24}(t,x) \leq \Psi(x - c_\varepsilon t), \qquad \int_0^\infty \Psi(\xi) d\xi < \infty. \tag{28}$$

This, together with (22), (26) and (27) finishes the proof of Lemma 3.□

Acknowledgement. We thank N. Vladimirova for discussions regarding numerical studies related to this work. This work was partially supported by ASCI Flash Center at the University of Chicago. AK was supported by NSF grant DMS-9801530, and LR was supported by NSF grant DMS-9971742.

References

[1] M. Abel, A. Celani, D. Vergni and A. Vulpiani, Front propagation in cellular flow, Preprint, 2000.

[2] B. Audoly, H. Berestycki and Y. Pomeau, Rèaction diffusion en ècoulement stationnaire rapide, C.R.Acad. Sci., Ser. IIB, **328**, 255-262.

[3] H. Berestycki and F. Hamel, Front propagation in periodic media, Preprint, 2000.

[4] H. Berestycki and L. Nirenberg, Traveling fronts in cylinders, Annales de l'IHP, Analyse non linéare, **9**, 1992, 497-572.

[5] A. Bonnet, Non-uniqueness for flame propagation when the Lewis number is less than 1, European J. Appl. Math. **6**, 1995, 287–306.

[6] P. Collet and J. Xin, Global existence and large time asymptotic bounds of L^∞ solutions of thermal diffusive combustion systems on $\mathbb{R}^n$, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **23**, 1996, 625–642.

[7] P. Constantin, A. Kiselev, A. Oberman and L. Ryzhik, Bulk burning rate in passive-reactive diffusion, Arch. Rat. Mech. Anal., **154**, 2000, 53-91.

[8] P. Constantin, A. Kiselev and L.Ryzhik, Quenching of flames by fluid advection, Preprint, 2000.

[9] R. Fisher, The wave of advance of advantageous genes, Ann. Eugenics, **7**, 1937, 355–369.

[10] M. Freidlin, Functional integration and partial differential equations, Annals of Mathematics Studies, **109**, Princeton University Press, Princeton, 1985.

[11] M. Freidlin and J. Gärtner, On the propagation of concentration waves in periodic and random media, Soviet Math. Dokl., **20**, 1979, 1282–1286.

[12] F. Hamel, Formules min-max pour les vitesses d'ondes progressives multidimensionnelles, Ann. Fac. Sci. Toulouse, **8**, 1999, 259-280.

[13] S. Heinze, G. Papanicolaou and A. Stevens, Variational principles for propagation speeds in inhomogeneous media, SIAM J. Appl. Math., to appear.

[14] A. Kiselev and L. Ryzhik, *Enhancement of the Traveling Front Speeds in Reaction-Diffusion Equations with Advection*, Annales del'IHP, Analyse non linéare, to appear.

[15] A.N. Kolmogorov, I.G. Petrovskii and N.S. Piskunov, *Étude de l'équation de la chaleurde matiére et son application à un problème biologique,* Bull. Moskov. Gos. Univ. Mat. Mekh. **1** (1937), 1–25.

[16] A. Majda and P. Souganidis, Large scale front dynamics for turbulent reaction-diffusion equations with separated velocity scales, Nonlinearity, **7**, 1994, 1–30.

[17] J. R. Norris, Long-time behavior of heat flow: global estimates and exact asymptotics, Arch. Rat. Mech. Anal., **140**, 1997, 161-195.

[18] J. Xin, Existence and nonexistence of traveling waves and reaction-diffusion front propagation in periodic media, Jour. Stat. Phys., **73**, 1993, 893-926.

[19] J. Xin, Analysis and modeling of front propagation in heterogeneous media, SIAM Review, **42**, 2000, 161-230.